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Application of the homotopy analysis method to determine the analytical limit state functions and reliability index for large deflection of a cantilever beam subjected to static co-planar loading

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ABSTRACT

In this paper, the Homotopy Analysis Method (HAM) is applied to obtain the limit state function, probability of failure and reliability index based on all stochastic and deterministic variables for a cantilever beam subjected to co-planar loading for the first time. First, it is established that a few iterations in the series expansion are sufficient to obtain highly accurate results and a substantial convergence region. After showing the effectiveness of HAM, two limit state functions are introduced as the maximum deflection in the y direction and maximum allowable stress, respectively. Then the first order reliability method (FORM) is employed to obtain reliability index, and omission sensitivity factor analytically. It is shown that HAM is a promising tool to obtain limit state function, probability of failure and reliability index analytically for nonlinear problems. Finally, a sensitivity analysis is done to show that which parameters could be considered deterministic or stochastic variables.

1. Introduction

The fundamental role of probability theory in safety and performance analysis is widely recognized in all branches of engineering. Probability theory provides a more accurate engineering representation of reality [1]. Reliability-based techniques allow designers to rationally assess the possibility of structural failure. It involves the use of probability and definition of a safety index to achieve a balance between safety and cost [2]. The traditional approach, the so-called "deterministic design", makes use of safety coefficients in order to prevent unpredicted failures due to the variability of the data [3]. The performance or integrity of a structure or structural assembly is generally evaluated by means of evaluating one or more variables such as the maximum displacement of a point, or the maximum stress or strain. When conducting such evaluations of performance or structural integrity many uncertainties exist such as e.g. unpredictability of loading conditions, inability to express the material properties accurately, simplifications in the modelling of the behaviour of the structure, limitations in the numerical methods, human errors or omissions, etc. Consequently, 100% reliability cannot be guaranteed, but the design can be conducted in order to raise the reliability up to a chosen level [4].

The starting point for all the methods which are used in reliability analysis is a performance function, which gives the relation between the chosen performance and the inputs of the model. A failure function can be expressed analytically only for simple problems, and generally it is given numerically, as it happens for industrial applications, where the finite element method is often employed for the analysis of structures. Hosseini and Khadem [5] studied the vibration and reliability

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Nomenclature

E	Young's modulus
g	Failure function
H	Auxiliary function
$\hbar$	Auxiliary parameter
I	Area moment of inertia
L	Undeformed length
P_A	Compressive force
Q_A	Transverse force
$R_m(U_{m-1})$	Remainder term
s	Arc length parameter
U	Normalized variables
x	Cartesian coordinate—beam length direction
X	Stochastic variables
x_A	x at beam end
y	Cartesian coordinate—beam thickness direction
y_A	y at beam end
β	Reliability index
γ_x	Maximum allowable deflection in the x direction
γ_y	Maximum allowable deflection in the y direction
ς_i	Omission sensitivity factor
η	Slope parameter
σ	Standard deviation
μ	Expected value
θ	Slope of normal to beam cross section relative to x axis
θ_A	Normal slope at the end section
λ	Proportionality factor.

of a rotating beam with random properties under random excitation using the finite element method. Renjian et al. [6] investigated the reliability evaluation of reinforced concrete beams by considering an elliptical function for combined shear–torsion for a linear case. Chandrasekhar and Sharma [7] analysed the reliability of a continuous beam. Givli and Altus [8] showed the effect of strength–modulus correlation on reliability of linearly elastic, brittle and stochastically heterogeneous beams by a functional perturbation method. Generally, analytical techniques have been used only for very simple cases, whereas more complex problems including nonlinearity have been addressed by application of numerical methods [9–13].

Most scientific problems in solid mechanics are inherently nonlinear by nature, and, except for a limited number of cases, most of them do not have analytical solutions. Accordingly, the nonlinear equations are usually solved using other methods including numerical techniques or by using analytical perturbation methods [14]. Therefore, obtaining analytical limit state functions or using analytical techniques to obtain reliability index for nonlinear problems is almost impossible. One of the semi-exact methods which do not need small/large parameters is the so-called Homotopy Analysis Method (HAM), first proposed by Liao in 1992 [15,16]. This method has already been applied successfully to solve many complex problems in solid mechanics as well in fluid mechanics [17–30].

In this paper, the Homotopy Analysis Method (HAM) is applied for the first time to analytically obtain the limit state function and reliability index for a geometrically nonlinear cantilever beam problem based on all stochastic and deterministic variables. First, it is shown that a few iterations in the series expansion are sufficient to obtain highly accurate results and a significant convergence region. Then the failure function is obtained as the maximum deflections in the y direction, and maximum strength, respectively. The length of the cantilever beam is considered as a deterministic variable, whereas Young's modulus, the area moment of inertia, and the shear and compressive resultants are considered as stochastic variables. By using the obtained analytical solution, the reliability index and omission sensitivity factors are computed based on different stochastic and deterministic variables for several examples. Finally, is calculated and the error of considering a parameter as deterministic is computed, and it is shown which parameters can be considered as deterministic or stochastic variables.

2. Mathematical formulation

A cantilever beam OA is subjected to co-planar loading consisting of an axial compressive force P_A and of a transverse force Q_A (Fig. 1). P_A and Q_A are follower forces, i.e., they will rotate with the end Section A of the beam during the deformation, and they will at all times remain tangential and perpendicular, respectively, to the deformed beam axis. Therefore, at any

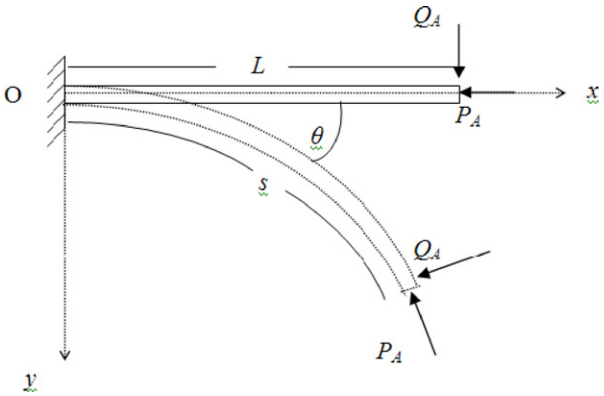

Fig. 1. The geometry and boundary conditions of a cantilever beam subjected to nonconservative external loading (follower forces).

point of coordinates $x(s)$, $y(s)$ the external moment M is expressed as [31]:

$$M = (P_A \cos\theta_A + Q_A \sin\theta_A)(y_A - y) + (-P_A \sin\theta_A + Q_A \cos\theta_A)(x_A - y) \tag{1}$$

where x, y are the longitudinal and transverse coordinates, respectively, q is the slope of the normal to the beam cross section, and x_A, y_A and θ_A denote the coordinates and the normal slope at the end section. The classical Euler–Bernoulli hypothesis assumes that the bending moment M at any point of the beam is proportional to the corresponding curvature [31,32], i.e.

$$M = EI\theta' \tag{2}$$

where E is Young's modulus, and I is the area moment of inertia of the beam cross section about the x axis. By using the following relations:

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta \tag{3}$$

and based on the trigonometric relations and by substituting Eqs. (2)–(5) into Eq. (1), the nonlinear differential equation governing the problem is obtained as follow:

$$\frac{\partial^2\eta(s)}{\partial s^2} + \frac{P_A}{EI}\sin(\eta(s)) + \frac{Q_A}{EI}\cos(\eta(s)) = 0. \tag{4}$$

The boundary conditions associated with the above equation are [31]:

$$\eta(0) = 0, \quad \eta'(L) = 0 \tag{5}$$

where

$$\eta = \theta - \theta_A. \tag{6}$$

Using only the two terms of a Taylor's series expansion for $\cos(\eta(s; q))$ and $\sin(\eta(s; q))$, and substituting in Eq. (1) yields:

$$\frac{\partial^2\eta(s)}{\partial s^2} + \frac{P_A}{EI}\left(\eta(s) - \frac{1}{6}\eta^3(s)\right) + \frac{Q_A}{EI}\left(1 - \frac{1}{2}\eta^2(s)\right) = 0. \tag{7}$$

To show the accuracy of the above assumption, a comparison is made between Eqs. (4) and (7) as shown in Fig. 2.

3. Application of HAM

In the following, the application of HAM to solve the defined problem is briefly explained. The nonlinear operator is defined as follows [31]:

$$N[\eta(s; q)] = \frac{\partial^2\eta(s; q)}{\partial s^2} + \frac{P_A}{EI}\left(\eta(s; q) - \frac{1}{6}\eta^3(s; q)\right) + \frac{Q_A}{EI}\left(1 - \frac{1}{2}\eta^2(s; q)\right) \tag{8}$$

where $q \in [0, 1]$ is the embedding parameter.

Expanding $\eta(s; q)$ in Taylor series with respect to q yields:

$$\eta(s; q) = \eta_0(s) + \sum_{m=1}^{\infty} \eta_m(s) q^m \tag{9}$$

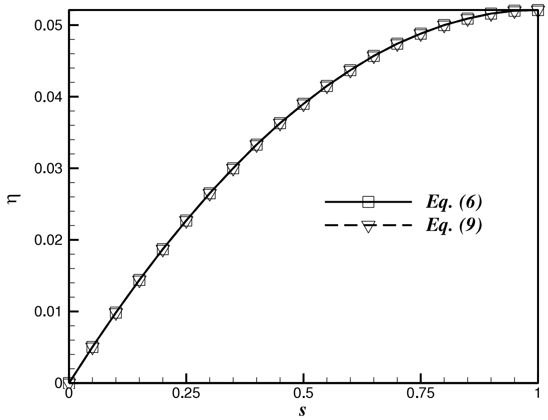

Fig. 2. Comparison between Eqs. (4) and (7).

where

$$\eta_m(s) = \frac{1}{m!}\frac{\partial^m \eta(s;q)}{\partial q^m}\bigg|_{q=0}. \tag{10}$$

The Homotopy Analysis Method can be adopted using many different base functions depending on the governing equations and the form of the flexural rigidity and Young's modulus. However, a set of base functions in the form can be used [31]:

$$\eta(s) = \sum_{n=0}^{\infty} b_n s^n \tag{11}$$

where b_n is a set of coefficients to be determined. In addition to the set of base functions, the auxiliary function $H(s)$, the initial approximation $\eta_0(s)$, and the auxiliary linear operator £ must be chosen in such a way that all solutions to the corresponding mth-order deformation equations exist and can be expressed by this set of base functions, whereas other expressions such as $s^n \sin\ (ms)$ must be avoided. This provides a so-called *rule of solution expression* [15]. By choosing the linear operator as:

$$£\left[\eta(s;q)\right] = \frac{\partial^2 \eta(s;q)}{\partial s^2} \tag{12}$$

the initial guess can be obtained as:

$$\eta_0(s) = 0. \tag{13}$$

The zeroth order deformation equation is:

$$\begin{aligned} &(1-q)\,L\left[\eta(s;q) - \eta_0(s)\right] = qhH(s)\,N\left[\eta(s;q)\right] \\ &\eta(l;q) = 0, \qquad \frac{\partial \eta(l;q)}{\partial s} = 0. \end{aligned} \tag{14}$$

Differentiating Eq. (8) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by $m!$, the so-called mth-order deformation equation for is $m \geq 1$ obtained.

$$\begin{aligned} &\eta_m(s) = \chi_m \eta_{m-1}(s) + \hbar \int_0^s \int_0^\tau H(s) R_m(\eta_{m-1}) ds\, d\tau \ + \ c_1 s + c_2 \\ &\eta_m(l) = \eta_m'(l) = 0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} R_m(\eta_{m-1}(s)) \, = \, &\left(\frac{d^2}{dq^2}\eta_{m-1}(s)\right) + \frac{P_A}{EI}\left(\eta_{m-1}(s) - \frac{1}{6}\sum_{j=0}^{m-1}\eta_{m-1-j}(s)\left(\sum_{z=0}^{j}\eta_{j-z}(s)\eta_z(s)\right)\right) \\ &+ \frac{Q_A}{EI}\left(1 - \frac{1}{2}\sum_{j=0}^{m-1}\eta_{m-1-j}(s)\eta_j(s)\cos(s-l)\right). \end{aligned} \tag{16}$$

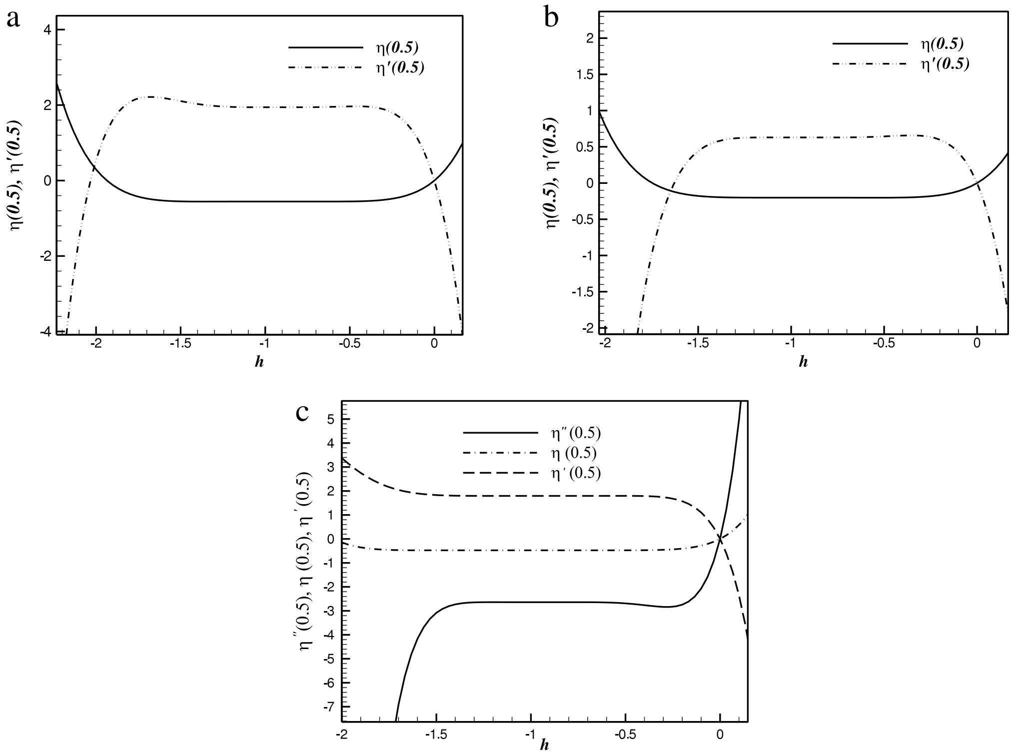

Fig. 3. The convergence region, $E = 10^9$, $I = 10^{-4}$, $L = 1$. (a) $Q_A = 50$ kN, $P_A = 100$ kN (b) $Q_A = 200$ kN, $P_A = 100$ kN, and (c) $Q_A = 400$ kN, $P_A = 200$ kN.

Here the auxiliary function was determined uniquely by $H(s) = 1$. Therefore the results are obtained as follows:

$$\begin{aligned}
&\eta_0(s) = 0\\
&\eta_1(s) = \frac{0.5hQ_A}{EI}(s^2 - 2Ls + L^2)\\
&\eta_2(s) = \frac{1}{EI^2}\left(2hQ_A\left(\begin{matrix}0.25s^2EI - 0.5LsEI + 0.02084P_Ahs^4 - 0.0834P_AhLs^3\\ +0.25P_Ahs^2EI + 0.125P_AhL^2 + 0.25L^2EI - 0.05shLEI+\\ 0.02084P_AhL^4 + 0.25hL^2EI - 0.0834P_AhsL^4\end{matrix}\right)\right)\\
&\vdots
\end{aligned} \tag{17}$$

The solution has been developed up to 6th order of approximation of $\eta(s)$.

4. Convergence of HAM solution

Obviously, the analytical solution should converge. It should be noted that the auxiliary parameter h, as pointed out by Liao [15], controls the convergence and accuracy of the series solution. In order to define a region such that the solution series is independent of h, a multiple of h-curves are plotted. The region where the distribution of η and η' versus h is a horizontal line is known as the convergence region for the corresponding function. In Fig. 3, the convergence region has been presented by considering the first six terms of the series solution. In addition, to validate the accuracy of the solution the results have been compared with numerical solution results as shown in the Table 1 for varying values of the applied loading, Young's modulus, area moment of inertia and beam length. The numerical solution results for the formulated initial value problem were obtained using the Maple 11 software. (The *dsolve* command with the numeric or *type = numeric* option on a real-valued two-point boundary value problem (BVP) finds a numerical solution for the ODE or ODE system BVP.) As it can be seen, for the mean values of the loads, Young's modulus and moment of inertia, $h = 1$ should be chosen.

5. Limit state function and reliability index

It is assumed that the stochastic variables X_i, $i = 1, \ldots, n$, are independent. Limit state functions are obtained by using HAM and FORM is employed to obtain reliability index. First, a transformation from X to a set of stochastic variables U that are normalized is defined [33].

Table 1
Comparison between HAM and numerical solution obtained by Maple 11 for different loading values (Q_A, P_A), Young's modulus (E), area moment of inertia (I) and beam length (L).

s	Q_A	P_A	E	I	η_{HAM}	η_{numeric}	η'_{HAM}	η'_{Numeric}
0.50	1.0E+06	1.0E+06	1.0E+09	1.0E−03	−0.50573	−0.50895	0.69120	0.69715
0.50	2.0E+06	1.0E+06	1.0E+09	1.0E−03	0.76824	0.77116	0.98030	0.99388
0.50	2.0E+06	1.0E+07	1.0E+09	1.0E−03	−0.19295	−0.19304	−0.62980	−0.62980
0.25	2.0E+06	1.0E+07	1.0E+09	1.0E−03	−0.05302	−0.05308	−0.42973	−0.42992
0.25	5.0E+06	5.0E+06	1.0E+10	1.0E−02	0.01114	0.01114	0.03826	0.03826
0.75	5.0E+06	5.0E+06	1.0E+10	1.0E−02	0.02392	0.02392	0.01278	0.01280
0.50	1.0E+06	1.0E+06	1.0E+10	1.0E−02	0.35999	0.36038	0.47285	0.47356

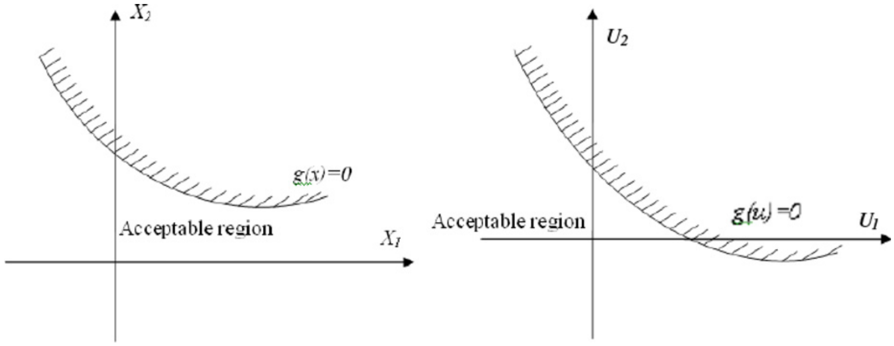

Fig. 4. Failure functions in the x-space and the u-space.

The failure function in the new U-space can be written as below (see Fig. 3):

$$g(U_i) = 0. \tag{18}$$

The reliability index β is defined as the smallest distance from the origin O in the u-space to the failure surface $g(U) = 0$. This is illustrated in Figs. 4 and 5. The point A on the failure surface closest to the origin (see Fig. 5) is denoted the β*-point* or the *design point* [32]. The reliability index is thus defined by the optimization problem [32]:

$$\beta = \min \sqrt{\sum_{i=1}^{n} u_i^2}. \tag{19}$$

Furthermore, the omission sensitivity factor ς_i is defined by:

$$\varsigma_i = \frac{1 - \alpha_i u_i^0/\beta}{\sqrt{1-\alpha_i^2}}. \tag{20}$$

If $u_i^0 = 0$ is chosen, then

$$\varsigma_i = \frac{1}{\sqrt{1-\alpha_i^2}}. \tag{21}$$

It should be mentioned that if $|\alpha_i| < 0.14$, then the error of the reliability index is less than 1% if a variable with $|\alpha_i| < 0.14$ is fixed.

The iteration scheme can be formulated to obtain the reliability index and omission coefficient factor (see the Appendix).

5.1. Failure function as the maximum beam deflection

In the first case, the failure function is taken as the maximum beam deflection. The maximum deflection in the y direction, must be computed. From [31] the following is obtained:

$$\theta(s) = \eta(s) + \theta_A \tag{22}$$

$$y_{\text{Max}} = y_A = \int_0^L \sin(\theta(s))\, ds. \tag{23}$$

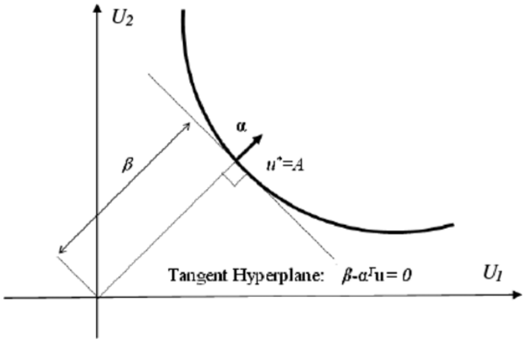

Fig. 5. Geometrical illustration of the reliability index, β.

Table 2
Stochastic parameters, distributions and mean values for the deflection limit state function.

Parameter	Distribution	Mean value (μ)
Shear force	*Gumbel*	450 kN
Normal force	*Gumbel*	400 kN
Moment of inertia	*Normal*	7.00E−05
Young's modulus	*Lognormal*	40 (GPa)

Table 3
Stochastic parameter, distribution and mean values for the stress limit state function.

Parameter	Distribution	Mean value (μ)
Shear force	*Gumbel*	350 kN
Normal force	*Gumbel*	300 kN%
Young's modulus	*Lognormal*	40 (GPa)
Length cross section	*Normal*	0.2 (m)
width cross section	*Normal*	0.1 (m)

The failure function is defined as follows:

$$g = \gamma_y - \frac{y_{\text{Max}}}{L} \geq 0 \tag{24}$$

where γ_y is maximum allowable deflection in the y direction. Here, the beam length L is considered as a deterministic variable, and the four stochastic variables are

$$X_1 = P_A, \qquad X_2 = Q_A, \qquad X_3 = I, \qquad X_4 = E. \tag{25}$$

The stochastic variables, mean values and coefficient of variation is shown it Table 2.

Therefore, the failure function can be expressed in the form:

$$g_y(P_A, Q_A, E, I) = g(U) = L\gamma_y - y_{\max} = 0. \tag{26}$$

5.2. Failure function as the maximum allowable stress

In the second case, the failure function is taken as the maximum allowable stress. The maximum allowable stress occurs at the root of the beam, as a result of bending and compression stress, therefore the cross section of the bean should be taken into account. The cross section is considered as a rectangle. Consequently, the area moment of inertia is divided into two independent parameters and the number of stochastic variables changes from 4 to 5. Thus, the area moment of inertia can be written:

$$I = \frac{1}{12} bh^3 \tag{27}$$

and the failure function can be presented in the form:

$$g(P_A, Q_A, E, b, h) = g(U) = 1 - \frac{\sigma}{\sigma_{\text{ultimate}}} = 0. \tag{28}$$

Stochastic parameters are shown in Table 3.

Table 4
Reliability index, unit vector, and omission sensitivity factor for the deflection limit state function where $\gamma_y = 0.25$.

Parameter	Symbol	COV		
Shear force	QA (kN)	10.00%	12.50%	15.00%
Normal force	PA (kN)	10.00%	12.50%	15.00%
Moment of inertia	I (m^4)	10.00%	12.50%	15.00%
Young's modulus	E (GPa)	3.00%	5.00%	7.00%
Reliability index	β	3.1294	2.9177	2.6083
Probability of failure	P_f	8.76E−04	1.77E−03	4.60E−03
α-vector	α_1	0.7428	0.8622	0.8871
	α_1	0.0268	0.0209	0.0196
	α_3	−0.6506	−0.4415	−0.3279
	α_4	−0.1557	−0.2475	−0.3244
Omission sensitivity factor	ζ_1	1.493622139	1.974069	2.166484
	ζ_2	1.000359314	1.000218	1.000192
	ζ_3	1.316793364	1.114503	1.058523
	ζ_4	1.012346181	1.032111	1.057172

Table 5
Reliability index, unit vector, and omission sensitivity factor for the ultimate strength limit state function.

Parameter	Symbol	COV	COV	COV
Shear force	QA (kN)	10.00%	12.50%	15.00%
Normal force	PA (kN)	10.00%	12.50%	15.00%
Length cross section	h (m)	10.00%	12.50%	15.00%
Width cross section	b (m)	10.00%	12.50%	15.00%
Young's modulus	E (GPa)	3.00%	5.00%	7.00%
Reliability index	β	3.419	3.094	2.715
Probability of failure	P_f	3.00E−04	1.00E−03	3.30E−03
α-vector	α_1	0.37	0.39	0.41
	α_2	0.03	0.03	0.03
	α_3	−0.78	−0.77	−0.77
	α_4	−0.5	−0.5	−0.49
	α_5	−0.02	−0.03	−0.04
Omission sensitivity factor	ζ_1	1.0764	1.0860	1.0964
	ζ_2	1.0005	1.0005	1.0005
	ζ_3	1.5980	1.5673	1.5673
	ζ_4	1.1547	1.1547	1.1472
	ζ_5	1.0002	1.0005	1.0008

6. Results and discussion

As mentioned before, the beam length L has been considered as a deterministic variable and in all cases $L = 1$. The remaining model parameters, including shear force Q_A, compressive force P_A, Young's modulus E, and the area moment of inertia I, are considered as stochastic variables. To show the application of HAM for computing the reliability index, the sensitivity factor and also the unit normal vector, two different cases have been considered.

In case I, as shown in Table 4, for different values of coefficient of variation, based on maximum value for the y-direction deflection ($\gamma_y = 0.25$), the reliability index and sensitivity factor have been computed. Model convergence was achieved within a few iterations for all cases. It is clearly observed that by increasing the coefficient of variation, the reliability index decreases, and as a consequence of this the sensitivity factor decreases and probability of failure increases. Based on the selected variables and from Eq. (21), it is concluded that the shear force, moment of inertia and Young's modulus should be considered as a stochastic variable, whereas the normal force can be considered as deterministic. Also, by increasing the coefficient of variation, the value of omission sensitivity factor increases for shear force and Young's modulus, which means the percentage of error increases if we consider them as deterministic variables and for two others decreases.

For case II, the same computing was done to calculate the reliability index and sensitivity factor for maximum allowable strength as shown in Table 5. From Table 5, it is observed that the reliability index decreases by increasing the value of coefficient of variation and also the omission sensitivity factor. It is seen that, shear force, width and length of cross section should be considered stochastic and the error of considering the normal force and Young's modulus deterministic is less than 1%. From the sensitivity analysis it is clear that the model is so sensitive to the cross section and the error of deterministic consideration h (length cross section) is more than 50%.

In this calculation, target reliability level was 3.1 with 10% coefficient of variation, but it was shown that by increasing the coefficient of variation, the reliability index decreases, as consequence of it probability of failure increases.

7. Conclusions

In this paper a HAM based solution procedure has been proposed to obtain the limit state function, reliability index and sensitivity factor for a Bernoulli–Euler cantilever beam loaded by circulatory forces. An excellent rate of convergence has been demonstrated, and the obtained results are in excellent agreement with results obtained from a numerical solution. It has been shown that HAM is an effective method to obtain the limit state function based on all the deterministic and stochastic variables for nonlinear problems, with the prerequisite that HAM can provide a general solution to the problem considered. After obtaining the failure function, the reliability index was computed, and the error of considering parameters as deterministic variables shown. From this, it can be concluded that the presented method is a convenient and efficient method for the reliability analysis for nonlinear problems (again with the prerequisite that HAM can provide a general solution to the problem considered), and that it can be used for wide range of loads and lengths for elastic beams with variable properties and undergoing large deformations.

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Appendix

The iteration scheme to obtain the reliability index and the omission sensitivity factor can be computed as follows:

(1) Guess u^0 and set $i = 0$.
(2) Calculate $g(U^i) = 0$.
(3) Calculate $\nabla g(U^i)$.
(4) Calculate an improved guess of the β point

$$u^{i+1} = \nabla g(u^i)\frac{\nabla g(u^i)^T u^i - g(u^i)}{\nabla g(u^i)^T \nabla g(u^i)}. \tag{A.1}$$

(5) Calculate the corresponding reliability index

$$\beta^{i+1} = \sqrt{(u^{i+1})^T u^{i+1}}. \tag{A.2}$$

(6) If convergence in β then stop, else $i = i + 1$ and go to 2.
A unit normal vector α to the failure surface at the β-point u^* is defined by:

$$\alpha = -\frac{\nabla g(u^*)}{|\nabla g(u^*)|}. \tag{A.3}$$

Therefore the β-point u^* can be written:

$$u^* = \beta\alpha. \tag{A.4}$$

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